Pumping Lemma for CFL

Let $L$ be a context free language. Then, there is a constant $n$, depending only on $L$, such that if $z$ is any string in $L$ and $|z| \geq n$, then we may write $z = uvwxy$ such that

1) $|vx| > 0$
2) $|vwx| \leq n$
3) $uv^iwx^iy$ is in $L$ for all $i \geq 0$. 
Applications of Pumping Lemma

- Pumping lemma for CFL is mainly used to prove some language is not a context free language.
  1. Select the language L you wish to prove non-CFL.
  2. Pick up an integer n.
  3. Select a string z in L. |z| must not be smaller than n.
  4. Break z into u, v, w, x, y in all possible ways so that |vwx| <= n and |vx| >= 1.
  5. Prove uv^iw^x^y for some i, for example 0, is not in L. From pumping lemma for CFL, L is not a context free language.
Example

1. Prove $a^ib^ic^i$ is not a CFL language.
2. Let’s consider the string $a^nb^nc^n$.
3. For any $u$, $v$, $w$, $x$, and $y$ such that $|vwx| \leq n$ and $|vx| \geq 1$.
   v and $x$ cannot contain both $a$ and $c$.
   v and $x$ contain a’s only, $uv^0wx^0y$ has fewer a’s than c’s.
   v and $x$ contains a’s and b’s, $uv^0wx^0y$ has fewer a’s and b’s than c’s.
   Similarly, we can show that for other combinations of $v$ and $x$, $uv^0wx^0y$ does not belong to $a^ib^ic^i$. 
Closure Properties of Regular Languages

• Theorem:
  If \( L \) is a context-free language over alphabet \( \Sigma \), and \( s \) is a substitution on \( \Sigma \) such that \( s(a) \) is a CFL for each \( a \) in \( \Sigma \), then \( s(L) \) is a CFL.

• Theorem:
  The context free languages are closed under union, concatenation, and Kleene closure.

• Theorem:
  The context free languages are not closed under intersection and complementation.

• Theorem:
  If \( L \) is a CFL and \( R \) is a regular language, then \( L \cap R \) is CFL

• Theorem:
  If \( L \) is a CFL and \( R \) is a regular language, then \( L - R \) is CFL
Decidable Problems

- If a CFL language is empty
- If a given string belongs to a CFL.
- If a CFL language is finite
- If a CFL language is infinite

- Equivalence of two CFL languages is not decidable.
Emptiness, Finiteness, and Infiniteness

- A CFL language is empty if and only if no terminal string can be derived from S.

- A CFL language is finite if and only if the graph constructed as described below has no cycle.
  - Convert the context free grammar to CNF with no useless symbols and $\varepsilon$-productions
  - Each variable is a vertex
  - There is an edge from A to B if there is a production of the form A->$BC$ or A->$CB$. 