Pushdown Automata

• A pushdown Automata is a machine which accepts a context-free language.

• Every context-free language is accepted by some pushdown automaton.

• A pushdown automaton is essentially a finite automaton with a stack (infinite memory space).
Pushdown Automata

Tape

Finite Control

Stack
Pushdown Automata: Example

- $L = \{wcw^R | w \in (0+1)^*\}$
- Two states: $q_1$ and $q_2$. $q_1$ is the initial state.
  1. If current input symbol is 0 and the current state is $q_1$, push 0 to the stack and stay in $q_1$.
  2. If current input symbol is 1 and the current state is $q_1$, push 1 to the stack and stay in $q_1$.
  3. If current input symbol is c and the current state is $q_1$, move to $q_2$.
  4. If current input symbol is 0 and the current state is $q_2$ and the top of the stack is 0, remove the top element of the stack and stay in $q_2$.
  5. If current input symbol is 1 and the current state is $q_2$ and the top of the stack is 1, remove the top element of the stack and stay in $q_2$.
  6. If both the stack and the input are empty, the string is accepted.
Pushdown Automata: Definition

- A pushdown automaton $M$ is a system $(Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$
  - $Q = \{q_0, q_1, \ldots, q_n\}$ is a finite set of states
  - $\Sigma$ is a finite input alphabet
  - $\Gamma$ is a finite stack alphabet
  - $\delta$ is a transition function mapping $Q \times (\Sigma \cup \{\varepsilon\}) \times (\Gamma \cup \{\varepsilon\})$ to finite subsets of $Q \times \Gamma^*$.
  - $q_0$ is the initial state.
  - $Z_0$ is the start symbol in the stack.
  - $F$ is the set of final states.
Moves

\[ \delta(q, a, Z) = \{(p_1, \gamma_1), (p_1, \gamma_1), \ldots, (p_m, \gamma_\mu)\}\]

q and \(p_i\) are states, \(a\) is in \(\Sigma\) and \(Z\) is \(\Gamma\), \(\gamma_i\) is in \(\Gamma^*\).

This move means that if the current state is \(q\), the current input symbol is \(a\), and the top stack symbol is \(Z\), then move to state \(p_i\), replace \(Z\) by \(\gamma_i\), and advance to the next input symbol.

\[ \delta(q, \varepsilon, Z) \] means without advancing to next input symbol.

\[ \delta(q, a, \varepsilon) \] means that the stack is empty.
Example: \( \{\text{wcw}^R|w \text{ in } (0+1)^*\} \)

- \( M = (\{q_0, q_1\}, \{0, 1\}, \{A, B\}, \delta, q_0, Z_0, \{\}) \)
  - \( \delta(q_0, 0, Z_0) = \{(q_0, A Z_0)\} \)
  - \( \delta(q_0, 0, A) = \{(q_0, AA)\} \)
  - \( \delta(q_0, 0, B) = \{(q_0, AB)\} \)
  - \( \delta(q_0, 1, Z_0) = \{(q_0, B Z_0)\} \)
  - \( \delta(q_0, 1, A) = \{(q_0, BA)\} \)
  - \( \delta(q_0, 1, B) = \{(q_0, BB)\} \)
  - \( \delta(q_0, c, Z_0) = \{(q_1, Z_0)\} \)
  - \( \delta(q_0, c, A) = \{(q_1, A)\} \)
  - \( \delta(q_0, c, B) = \{(q_1, B)\} \)
Accepted Language

• By final states
  \( w \) is in \( L(M) \) if there exists a sequence of moves from \((q_0, w, Z_0)\) to \((p, \varepsilon, \gamma)\) for some \( p \) in \( F \) and \( \gamma \) in \( \Gamma^* \).

• By empty stack.
  – \( w \) is in \( L(M) \) if there exists a sequence of moves from \((q_0, w, Z_0)\) to \((p, \varepsilon, Z_0)\) for some \( p \) in \( Q \).
Pushdown Automata And Context-Free Grammars

- For every context-free grammar $G$, there is a pushdown automaton $M$ such that $L(M) = L(G)$.

- For every pushdown automaton $M$ there is a context-free grammar $G$ such that $L(G) = L(M)$. 
Convert CFG to PDA

• Given a CFG \( G = (V, T, P, S) \), we construct the PDA as \( M = (Q, \Sigma, \Gamma, \delta, s, Z_0, \{ q \}) \)
  
  - \( Q = \{ s, t \} \)
  
  - \( \delta(s, \varepsilon, Z_0) = (q, S) \)
  
  - for every \( a \) in \( T \), \( \delta(q, a, a) = \{(q, \varepsilon)\} \)
  
  - for every rule \( A \rightarrow \alpha \), where \( \alpha \) is in \((V+T)^*\), \( \delta(q, \varepsilon, A) = \{(q, \alpha)\} \)
Example

- $S \rightarrow aS | aSbS | \varepsilon$

- $\delta(s, \varepsilon, Z_0) = (q, S)$
- $\delta(q, a, a) = (q, \varepsilon)$
- $\delta(q, b, b) = (q, \varepsilon)$
- $\delta(q, \varepsilon, S) = \{(q, aS) (q, aSbS), (q, \varepsilon)\}$